A STOCHASTIC MODELING FOR THE UNSTABLE FINANCIAL MARKETS

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Abstract: : An alternative approach to stochastic calculus for a financial model on some imperfect and unstable financial markets is proposed. Following the most recent instrument for the financial modeling, we study the solvability of a class of forward-backward stochastic differential equations (FBSDE) in the framework of McShane stochastic calculus, in some general hypothesis on the initial value and the coefficient functions.

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1. INTRODUCTION

Continuous development of the finance assets yields more adequate mathematical models which are supposed be good enough to describe the complex behavior of the financial market. More sophisticated models are available on some restrictive financial hypothesis, but this hypothesis are not satisfied on some transition financial markets as in East Europe, where are more unpublished information, over quoted initial values and government financial interventions. Moreover, the evolution on these markets is characterized by some "smoothed" life-time and some very "noises" life-time and this time periods are hard unexpected. For these reasons, we propose an approach somehow, more general as there for a free financial market.

The classic stochastic approach for the financial models has used the framework developed by Ito to deal with the resulting stochastic differential equations (SDE), based on the idea that a Wiener stochastic process is used for the external disturbances. Then, more authors supposed an semimartingale process for the external noises which make very complicated stochastic calculus. On the other side, E.J.McShane developed a more simple integration calculus using the Ito-belated integrals. In 1979, Ph.Protter showed that the McShane calculus is equivalent with the integration with respect to a semimartingale process. Somehow, this situation is similar with the fact that a Riemann-Stieltjes integral can be considered as a Lebesque integral in some adequate framework, but practically we prefer to use the Riemann integration calculus as to be more simple.

It's known that the stochastic approach for the financial modeling is started with the famous papers of Merton and Black and Scholes in the 70's. The principal instrument for the stochastic modeling is the backward stochastic differential equations (BSDE) or forward-backward stochastic differential equations (FBSDE). This leads us to consider a forward-backward McShane stochastic differential equations.

2. PRELIMINARY RESULTS

In first year of 70's, E.J.McShane introduced so called belated integrals and stochastic differentials and differential systems which enjoying the following three properties: inclusiveness, consistency and stability. McShane's calculus had proved to very valuable in modeling and is finding applications in physics, engineering and economics.

A stochastic integral equations by McShane type is one of the following form:

$$X(t) = x(0) + \int_0^t f(s, X(s))ds + \sum_{j=1}^{r}\int_0^t g_j(s, X(s))dz_j(s) + \\ + \sum_{j,k=1}^{r}\int_0^t h_{jk}(s, X(s))dz_j(s)dz_k(s) \qquad (1)$$

where the integrals are belated or McShane integrals.

On the above equation, we recall some specifically results of the McShane stochastic calculus. Let (Ω, F, P) be a complete probability space and let $\{F_t, 0 \le t \le a\}$ be a family of complete $\sigma-$subalgebras of F such that $0 \le s \le t \le a$ then $F_s \subseteq F_t$. Every process denoted by z with diferent affixes will be a real valued second order stochastic process adapted to $\{F_t, 0 \le t \le a\}$ (i.e. $z(t)$ is F_t-measurable for every $t \in [0, a]$) and

$$|E[(z(t) - z(s))^m / F_s]| \le K(t - s)$$

a.s., whenever $0 \le s \le t \le a, m = 1, 2, 4,$ for a positive constant K having a.s. continuous sample functions (and we say that the process satisfies a K-condition).

It is known (see [7]) that if $f : [0, a] \to L_2$ is a measurable process adapted to the F_t and if $t \to \| f(t) \|$ is Lebesgue integrable on $[0, a]$, then if z_1 and z_2 satisfy a K-condition, the McShane integrals $\int_0^a f(t)dz_1(t)$ and $\int_0^a f(t)dz_1(t)dz_2(t)$ exist and the following estimates are true

$$\| \int_0^a f(t)dz_1(t) \| \le C \{\int_0^a \| f(t) \|^2 dt\}^{\frac{1}{2}} \qquad (2)$$

$$\| \int_0^a f(t)dz_1(t)dz_2(t) \| \le C \{\int_0^a \| f(t) \|^2 dt\}^{\frac{1}{2}} \quad (3)$$

where $C = 2Ka^{\frac{1}{2}} + K^{\frac{1}{2}}$.

An important class of McShane stochastic differential equations is the class of equation which have a canonical extension or a canonical form (as in McShane a), i.e. the equation (1) with the special case when

$$h_{jk}(t, X(t)) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial g_j(t, X(t))}{\partial X^i}\cdot g_k(t, X(t)),\quad X = (X^1, \dots, X^n). \quad (4)$$

Among to this forward equations, in the optimal stochastic control appear some backward differential equations as the following

$$\begin{aligned} &Y(t) + \int_t^1 f(s, Y(s), Z(s))ds + \sum_{j=1}^{r}\int_t^1 [g_j(s, Y(s), Z(s))]dz_j(s,\omega) + \\ &+ \sum_{j,k=1}^{r}\int_t^1 [h_{jk}(s, Y(s), Z(s))]dz_j(s)dz_k(s) = Y_1 \end{aligned} \quad (5)$$

where $\{z_j(t), 0 \le t \le 1\}$, $j = 1, 2, \dots, r$ is a stochastic process defined on the probability space (Ω, F, P) with the natural filtration $\{F_t, 0 \le t \le 1\}$ and Y_1 is a given F_1-measurable random variable such that $E|Y_1|^2 < \infty$. Moreover, f is a mapping from $\Omega\times[0,1]\times R\times R$ to R which is assumed to be $P\otimes B\otimes B\setminus B$-measurable, where P is the σ-algebra of F_t-progressively measurable subsets of $\Omega\times[0,1]$. Also g is a mapping from $\Omega\times[0,1]\times R$ to R which is assumed to be $P\times B\setminus B$-measurable.

We remark that in the case of backward stochastic differential equations by the McShane type we have a canonical extension when replace the functions h_{jk} as above.

In this context we consider the following forward-backward stochastic differential equation by the McShane type

$$\begin{cases} X_t = X_0 + \sum_{j=1}^{r}\int_0^t a_j(s, X(s), Y(s), Z(s))dz_j(s) + \\ \quad + \sum_{j,k=1}^{r}\int_0^t b_{jk}(s, X(s), Y(s), Z(s))dz_j(s)dz_k(s) \\ Y_t = \sum_{j=1}^{r}\int_t^1 f_j(s, X(s), Y(s), Z(s))dz_j(s) + \\ \quad + \sum_{j,k=1}^{r}\int_t^1 g_{j,k}(s, X(s), Y(s))dz_j(s)dz_k(s) - h(Y_1) \end{cases} \quad (6)$$

with $a, b, f, g: \Omega\times(0,1)\times R\times R\times R \to R$, $h: R^+ \to R$ and the following hypotheses (which extend the result of Athanassov 1990 [1] for ordinary differential equations and includes other results on FBSDE):

i) a, b, f and g is $P \otimes B \otimes B \otimes B$ measurable functions;

ii) $\varphi(\cdot, 0, 0, 0) \in M^2((0,1), R)$, where φ is any functions a, b, f or g ($M^2(0,1)$ is the set of all stochastic process which are square McShane integrable on $[0,1]$ and F_t-measurable for $0 \le t \le 1$);

iii) there exists $u(t)$ a continuous, positive and derivable function on $0 < t \le 1$ with $u(0) = 0$, having nonnegative derivative $u'(t) \in L([0,1])$, with $u'(t) \to \infty, t \to 0^+$ such that

$$| \varphi(t, x_1, y_1, z_1) - \varphi(t, x_2, y_2, z_2) |^2 \le \frac{u'(t)}{Ku(t)} \min(| x_1 - x_2 |^2, | y_1 - y_2 |^2, | z_1 - z_2 |^2),$$

$$| h(y_1) - h(y_2) | \le \frac{u'(t)}{Ku(t)} | y_1 - y_2 |^2, \qquad (7)$$

for all $x_1, x_2, y_1, y_2, z_1, z_2 \in R, 0 \le t \le 1$, positive constant K and φ is any function a, b, f or g;

iv) with the same functions $u(t)$ as above,

$$| \varphi(t, x, y) |^2 \le u'(t) \min(1 + | x |^2, 1 + | y |^2, 1 + | z |^2), \ \ and \ \ | h(y) | \le (1 + | y |^2), \quad (8)$$

and X_0 is a finite random variable and Y_1 is given F_1-measurable random variable such that $E | Y_1 |^2 < \infty$.

A similar forward-backward equation can be obtain using the canonical form..

3. Main results

In this section, we prove that the stochastic (forward-backward) differential system has a solution on some interval $[\delta, 1]$ for any positive constant δ in some general hypotheses of the coefficient functions a, b, f and g as above. This case, with some discontinuity in the initial time moment $t = 0$ is is in according situation of the transition financial markets, where the underlying assets (which is modeling with a forward stochastic differential equation) is over-quoted.

3..1 Existence and uniqueness

We have the following theorem:

Theorem 1. *Let be a, b, f, g and h satisfying the hypotheses $i) - iv)$ and $Y_1 \in L^2(\Omega, F_1, P, R)$, then there exists a triple $(X, Y, Z) \in$ $\in (M^2((0,1) \times M^2((0,1)) \times M^2((0,1)), R)$ which satisfy the system (6) in the canonical form, for $0 \le t \le 1$*

Proof: In a similar way as [14].

3.2. Option pricing

The valuation of contingent claims is proeminent in the theory of modern finances. Typical claims such as call and put options are significant not only in theory but in real security markets.

The option pricing model developed by Black and Scholes [2], formalized and extended in the same year by Merton [13], enjoys great popularity.

We consider a Black-Scholes market $M_{BS} = (S, B, \phi)$ (see [10],[14]) where:

i) $S = \{S_t\}$, $t \in [t_0, T]$, $t_0 \ge 0$ is the price process of a stock and we suppose that it satisfies the following differential stochastic equation by McShane type:

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dz_t + \rho(t, S_t)(dz_t)^2, \quad (9)$$

where

$$\mu(t, S_t) = \frac{-b}{1-\beta}\frac{S_t}{t}, \quad \sigma(t, S_t) = ct^b S_t^\beta, \quad \rho(t, S_t) = \frac{\beta}{2}c^2 t^{2b} S_t^{2\beta-1}, \quad (10)$$

with $-1 < b < 0$, $0 < \beta \le 1$, $c \in R$;

ii) $B = \{B_t\}$, $t \in [t_0, T]$ is the price process of a bond and we consider that it satisfies the differential stochastic equation by McShane type:

$$dB_t = rdt + l(dz_t)^2; \quad (11)$$

iii) ϕ is a trading strategy (see [14]) i.e. a pair $\phi = (\phi^1, \phi^2)$ of progressively measurable stochastic processes on the underlying probability space (Ω, F, P).

It is known (see [14]) that a trading strategy ϕ over the time interval $[t_0, T]$ is self-financing if its wealth proces $V(\phi)$, which is set equal

$$V_t(\phi) = \phi_t^1 S_t + \phi_t^2 B_t, \ \forall\, t \in [t_0, T]$$

satisfies the following condition

$$V_t(\phi) = V_{t_0}(\phi) + \int_{t_0}^{t} \phi_u^1 dS_u + \int_{t_0}^{t} \phi_u^2 dB_u, \ \forall\, t \in [t_0, T]$$

where the integrals are understood in the McShane sense.

Remark. We observe that the coefficient functions of the stochastic differential equation (5) satisfy the conditions of our Theorem, for

$$u(t) = \frac{1}{t^{2b}}, t \in (0, T].$$

We consider a European call option written on a stock S, with expiry date T and strike price K. Let the function $c : R_+ \times [t_0, T] \to R$ ($t_0 \ge 0$) given by the formula

$$c(s,t) = D(A(t)s - K)e^{B(t)s^\alpha + C(t)}, \quad (12)$$

where $A, B, C : [0, T] \to R$ are some continuous functions, D is a positive constant and $\alpha = -2\beta$.

In [Negrea 2003b, [11]] is proved the following results (using classical method of PDEs)

Theorem 2. *The arbitrage price at time* $t \in [t_0, T]$ *of the European call option with expiry date* T *and strike price* K *in the Black-Scholes market is given by the formula*

$$C_t = c(S_t, T-t), \forall\, t \in [t_0, T], \quad (13)$$

where the function $c: R_+ \times [t_0, T] \to R$ is given above and $t_0 \geq 0$.

Remark. It is easy to check that the formula (14) is true using the FBSDEs method (given in Theorem 1.).

4. SOME APPLICATIONS AND EXAMPLES

We consider that we have an European call option on a convertible currency (such type of derivative assets are transactioned on the Sibiu Monetary-Financial and Commodities Exchange). More specifically we have an option on the report EUR/RON (Euro/Romanian Leu) from 01.10.2009 to 30.04.2010. The behaviour of this process is given in the graph from bellow and we can sea more very smothed part of this simple path and this explain a non-random noise on the market (in fact these are the results of some financial policies of Romanian government).

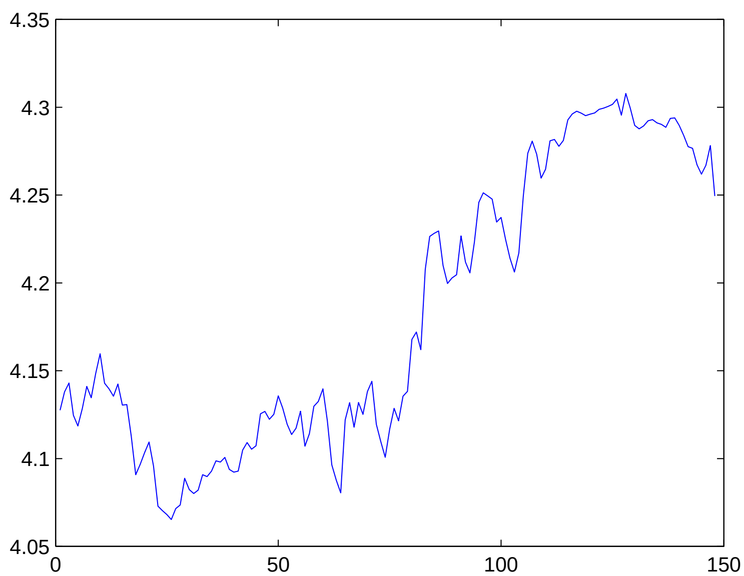

Figure no. 1

We compute the price with our formula and we obtain $C(t_0, T) = 0.768$ which is less than the price of the option at the Sibiu Monetary-Financial and Commodities Exchange (compute with classical formula of Black-Scholes), but our price is more closed to reality.

5. CONCLUSIONS

.We proposed a model for the behavior of the financial asstes on some unstable finacila maerkets. The evolution on these markets is characterized by some "smoothed" life-time and some very "noises" life-time and this time periods are hard unexpected. For these reasons, we propose an approach somehow, more general as there for a free financial market. Out study is just at the begining, but, as inthe example form above, the obtained results sustain our modeling for applications on the Romanian finacial market where the noise market is not a classical Gaussian noise, there exists more others random or non-random perturbations.

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